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A Non-Maxwellian Steady Distribution for One-Dimensional Granular Media

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We consider a nonlinear Fokker–Planck equation for a one-dimensional granular medium. This is a kinetic approximation of a system of nearly elastic particles in a thermal bath. We prove that homogeneous solutions tend asymptotically in time toward a unique non-Maxwellian stationary distribution.

KEY WORDS: Granular media; inelastic collisions, kinetic equations; Fokker–Planck.

1. INTRODUCTION

A simple model of granular media, widely studied in the last years, is a one-dimensional system of N particles colliding inelastically. In such a model the particles move freely between two consecutive collisions and in the collision instant the impulse is conserved while the kinetic energy is dissipated. The collision rule is:

$$v' = v_1 + \varepsilon(v - v_1), \qquad v'_1 = v - \varepsilon(v - v_1) \tag{1.1}$$

where ε is the inelasticity parameter and v', v'_1 and v, v_1 are the outgoing and ingoing velocities respectively.

This model has many interesting features. First of all it can deliver collapses. For $N = 3$ it is possible to prove that, if ε is larger than some critical value, there is a positive measure set of initial conditions for which the particles perform infinitely many collisions and converge to the same point in a finite time (see refs. 9 and 15). In refs. 4 and 15 the collapse for

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a large number of particles has been investigated and a conjecture on the minimum value of ε (given N) to have a collapse was formulated. In general it is possible to prove that, if $\varepsilon N < \log 2$, there is no collapse. Moreover if $\varepsilon N > \pi$, there exists an initial particle configuration leading to a collapse (see ref. 2).

Numerical simulations showing clustering of particles in dimension two are described in ref. 12.

A challenging problem is to give an hydrodynamic description of granular media (see, e.g., ref. 7, 13), and this model, thanks to its simplicity, is a natural candidate to check the validity of this kind of description.[10, 16, 17] In particular in ref. 17 it has been shown numerically, that an hydrodynamic description works if the equation are closed at the third moment of the velocity and not, as usual, at the second moment (the temperature). Moreover it has been outlined by Y. Du *et al.* (see ref. 10) anomalous thermodynamic and hydrodynamic behavior due to the tendency of the system to clusterize. In particular, putting the system in a slab and pumping energy from a wall at a constant temperature, they observe no energy equipartition: most of the particles are far from the wall and basically at rest.

Therefore, in order to compensate the strong tendency of the system to dissipate energy, it is natural to investigate the response of the system in a thermal reservoir. In this problem we face this problem in a kinetic approximation which we are going to explain.

Consider the limit $N \to \infty$, $\varepsilon \to 0$, $N\varepsilon \to \lambda$, see refs. 3, 10, and 16. In such a way it is possible to derive (formally) the following kinetic equation for the phase space distribution function f:

$$(\partial_t + v\,\partial_x)\, f = -\lambda\, \partial_v(Ff) \tag{1.2}$$

where

$$F(x, v, t) = \int \phi(\bar{v} - v)\, f(x, \bar{v}, t)\, d\bar{v} \tag{1.3}$$

and $\phi(y) = y\,|y|$, $y \in \mathbb{R}$. For this equation it is possible to prove existence of smooth solutions for a short time and of global solutions for small data.[3]

We mention that it is still unclear whether collapses, namely the formation of a δ-component, can occur in a finite time, is still open.

Incidentally we mention that other kinetic picture (Boltzmann like descriptions) are possible in dimension one,[10, 15] and larger.[11]

In this paper we consider an inelastic particle system in the kinetic picture (1.2), in a thermal bath at a constant temperature, described, as usual, by a Fokker–Planck term:

$$(\partial_t + v\,\partial_x)\,f = -\lambda\,\partial_v(Ff) + \beta\,\partial_v(vf) + \sigma\,\partial_v^2 f \tag{1.4}$$

We shall actually consider the much simpler homogeneous case: f is only a function of the velocity. We stress that this analysis is not academic. Indeed in view of a possible hydrodynamic limit for the system (1.4), it is interesting to investigate carefully what happens locally, when, in the fast thermalization scale, the homogeneous regime is dominant.

We shall prove that any solution of the homogeneous version of Eq. (1.4) converges, as $t \to \infty$, to a steady state which is not Maxwellian. Such stationary solution is described implicitly by an equation of mean-field type, (Eqs. (2.5)–(2.6) below) and, for large v, behaves as $\exp - C\,|v|^3$. In other words the inelastic interaction makes the distribution more picked around than the usual Maxwellian distribution.

We do not know whether this solution is stable for the general non homogeneous case. Also we do not know whether the thermal bath is sufficient to prevent the solution to have a collapse in a finite time. However we want to mention that a three-particles system in a thermal bath do deliver collapses for a positive set of initial conditions, (see ref. 2). As a consequence we conjecture that the Fokker–Planck term does not help too much to prevent singularities for the solutions to Eq. (1.4).

Let us finally remark that related asymptotics of Fokker–Planck type equations have been studied in refs. 1, 5, and 6.

2. CONVERGENCE TO EQUILIBRIUM

Consider the following initial value problem

$$\begin{gathered} \partial_t f + \partial_v[(\lambda F - \beta v)\,f] = \sigma\,\partial_v^2 f \\ f(v, 0) = f_0(v) \end{gathered} \tag{2.1}$$

where

$$F(v, t) = \int \phi(\bar{v} - v)\,f(\bar{v}, t)\,d\bar{v} = -(\phi * f(\cdot, t))(v) \tag{2.2}$$

Here $f = f(v, t)$ denotes the one-particle phase space distribution for a large system of inelastic particles in a homogeneous mean-field approximation: each particle can interact with any other, no matter where they are localized. Moreover the system is in contact with a thermal reservoir at temperature $T=\sigma/\beta$ modeled by the Fokker–Planck term in Eq. (2.1). Indeed, for $\lambda=0$ in Eq. (2.1), the only invariant measure of the system is the Maxwellian:

$$M(v)=Ce^{-\beta v^2/2\sigma} \tag{2.3}$$

where C is a normalization factor.

From now on we shall assume that the total momentum is vanishing at time zero (and hence at later times):

$$\int vf_0(v)\,dv=\int vf(v,t)\,dv=0 \tag{2.4}$$

In this paper we are interested in the limit $t\to\infty$ for the solutions to Eq. (2.1). A trivial calculation shows that the (formal) equilibria of Eq. (2.1), satisfy the following equation:

$$\bar{f}(v)=\frac{1}{Z}e^{-((\beta/2\sigma)v^2+(\lambda/3\sigma)\int|v-\bar{v}|^3\bar{f}(\bar{v})\,d\bar{v})} \tag{2.5}$$

where

$$Z=\int dv\,e^{-((\beta/2\sigma)v^2+(\lambda/3\sigma)\int|v-\bar{v}|^3\bar{f}(\bar{v})\,d\bar{v})} \tag{2.6}$$

We shall prove that there exists a solution to Eq. (2.5) which minimize a suitable free-energy functional (defined in Eq. (2.11) later on). Moreover the solutions of the initial value problem (2.1) converge to such steady solution. A remarkable fact of this analysis is that the above solution is not Maxwellian.

Before dealing with this asymptotics we first spend a few words about the existence of the solutions. We first notice that the Cauchy problem (2.1) can be characterized in terms of a nonlinear diffusion process (in the Mc Kean sense, see ref. 14). In fact, consider the following stochastic process:

$$dV(t,v)=-(\lambda\phi * f(t)(V(t,v))-\beta V(t,v))\,dt+\sqrt{2\sigma}\,db; \qquad v(0,v)=v \quad \text{a.e.} \tag{2.7}$$

where $f(t)$ is given by:

$$\int f(v,t)\,u(v)\,dv=\mathbb{E}\left(\int f_0(v)\,u(V(v,t))\,dv\right) \tag{2.8}$$

and $\mathbb{E}$ denotes the expectation, u a continuous test function and V is the solution of Eq. (2.7). The fixed point problem arising from (2.7) and (2.8) can be solved with some modifications as in ref. 3 where it was investigated the case $\sigma = \beta = 0$. In particular we can construct a classical solution to Eq. (2.1) provided that to $f_0 \in \mathbf{C}^2(\mathbb{R})$. More precisely it can be proved that, given $(1 + v^4) f_0 \in \mathbf{L}^1(\mathbb{R})$, there exists a unique solution of Eq. (2.1) (with this regularity) satisfying the nonlinear integral equation

$$f(t, v) = (M_{v(t)} * f_0)_{\alpha(t)} + \lambda \int_0^t e^{\beta s}(-\partial_v M_{v(t-s)} * (F(s) f(s))_{\alpha(s)^{-1}})_{\alpha(t)} \, ds$$

where $M_\nu(v)$ is the Maxwellian distribution of temperature $\nu > 0$,

$$(g)_\alpha (v) = \alpha^{-1/2} g(\alpha^{-1/2} v), \qquad v \in \mathbb{R}, \quad \alpha > 0$$

$\alpha(t) = e^{-2\beta t}$ and $\nu(t) = \sigma/\beta(e^{2\beta t} - 1)$.

Related existence results can be seen in refs. 8 and 18.

There is an important difference with the deterministic case we want to stress. For $\sigma = \beta = 0$ the solution becomes of compact support at any positive time and this describe the strong tendency of the system to concentrate around $v = 0$. Here (for $\sigma \neq 0$), the diffusion prevents this fact and $f(v, t) > 0$ for $t > 0$. However the moments of the distribution can be controlled easily.

For a positive even integer $p \geqslant 2$, we have:

$$\frac{d}{dt} \int v^p f(v, t) \, dv \leqslant -\beta \int v^p f(v, t) \, dv + \sigma p(p-1) \int v^{p-2} f(v, t) \, dv \qquad (2.9)$$

Inequality (2.9) is consequence of the fact that the nonlinear term (see ref. 3), gives rise to a negative contribution.

By (2.9) we use the mass conservation to control the kinetic energy uniformly in time. By a recursive argument we also have the bound:

$$\sup_{t \in \mathbb{R}} \int v^p f(v, t) \, dv \leqslant C \qquad (2.10)$$

provided that $\int v^p f_0(v) \, dv < +\infty$.

Let us now pass to analyze the asymptotic behavior of the solutions to Eq. (2.1). The basic tool in our analysis is the following free-energy functional:

$$\eta(f) = \int f(v) \log f(v) \, dv + \frac{\lambda}{6\sigma} \int |v - \bar{v}|^3 f(v) f(\bar{v}) \, dv \, d\bar{v} + \frac{\beta}{2\sigma} \int v^2 f(v) \, dv \qquad (2.11)$$

The formal variation of this functional (on the probability distribution densities) is:

$$\delta\eta = \int \left[\log f(v) + \frac{\lambda}{3\sigma} \int |v-\bar{v}|^3 f(\bar{v})\, d\bar{v} + \frac{\beta}{2\sigma} v^2 \right] \delta f(v)\, dv \tag{2.12}$$

As consequence $\delta\eta(\bar{f}) = 0$, implies $\bar{f}$ solution to Eq. (2.5).

Remark. Consider a particle discretization of our system given by the Langevin equation:

$$dv_i = \frac{\lambda}{N} \sum_j \phi(v_i - v_j)\, dt - \beta v_i\, dt + \sqrt{2\sigma}\, db_i, \qquad i = 1, 2,..., N \tag{R.1}$$

where b_i are N independent brownian motion.

Notice that the drift term in the right-hand side of (R.1) is the gradient of the function:

$$\frac{\beta}{2} \sum_i v_i^2 + \frac{\lambda}{6N} \sum_{i,j} |v_i - v_j|^3 \tag{R.2}$$

Therefore the process solution of (R.1) has a unique invariant measure given by

$$const.e^{-(1/2\sigma)[\sum_{i=1}^N \beta v_i^2 + (\lambda/6N) \sum_{i,j} |v_i - v_j|^3]} \tag{R.3}$$

In the mean field limit $N \to \infty$ the one-particle distribution function relative to distribution (R.3) converges to the solution of the self-consistent equation (2.5). This argument suggests the introduction of the functional (2.11). Eq. (2.5) can be obtained as a mean-field limit of the equilibrium distribution for a system of N inelastic particles, with $\varepsilon N = \lambda$. Moreover, Eqs. (2.7)–(2.8) are a nonlinear Langevin equation with potential given by $(\beta/2)v^2 + (\lambda/3) \int d\bar{v}\, |v - \bar{v}|^3 f(\bar{v})$. This formally implies that the equilibrium is given by Eq. (2.5), and that η, as in (2.11), is a Lyapunov function.

Theorem 2.1. Consider the functional η defined by (2.11) on the set:

$$\mathscr{P} = \left\{ f \in \mathbf{L}_1(\mathbb{R}) \mid f \geqslant 0, \int f = 1, \int v f(v) = 0 \right\} \tag{2.13}$$

Then η has a unique minimum, $\bar{f}$, in $\mathscr{P}$, and $\bar{f} \in \mathbf{C}^\infty$ and satisfies (2.5).

Proof. We first show that η is bounded from below. Denoting by $\chi(A)$ the characteristic function of the set A, we have, for any $f \in \mathscr{P}$:

$$\begin{aligned}
-\int f \log f\chi(f<1)\,dv &= -\int f \log f\chi(e^{-\beta/\sigma\,|v|}<f<1))\,dv \\
&\quad -\int f \log f\chi(f<e^{-\beta/\sigma\,|v|})\,dv \\
&\leqslant \frac{\beta}{\sigma}\int |v|\,f(v)\,dv - \int \sqrt{f}\log f e^{-\beta/2\sigma\,|v|}\chi(f<1)\,dv \\
&\leqslant \frac{\beta}{2\sigma}+\frac{\beta}{2\sigma}\int v^2 f(v)\,dv + M\,\frac{4\sigma}{\beta}
\end{aligned} \tag{2.14}$$

where

$$M = \sup_{0<r<1} -\sqrt{r}\log r$$

Therefore $\eta \geqslant -(\beta/2\sigma + 4M\sigma/\beta)$.

Consider now a minimizing sequence $\{f_n\}_{n=1}^{\infty}$:

$$\bar{\eta} = \inf_{f\in\mathscr{P}} \eta(f) = \lim_{n\to+\infty} \eta(f_n)$$

Note that the three terms, entropy, interaction term and kinetic energy, computed on the sequence f_n, must be uniformly bounded, and then the third moment of f_n is also uniformly bounded.

Extract now a converging subsequence (still denoted by f_n) in the sense of the weak convergence of the measure. Let $\bar{f}$ be the limit. Notice that $\bar{f}$ is a probability density by the entropy and energy control. Moreover the entropy control implies that the convergence holds weakly in $\mathbf{L}_1$. By the lower semi-continuity of the entropy:

$$\int \bar{f}\log\bar{f} \leqslant \liminf_{n\to\infty}\int f_n \log f_n$$

Moreover

$$\int |v-\bar{v}|^3\,\bar{f}(v)\,\bar{f}(\bar{v})\,dv\,d\bar{v} \leqslant \liminf_{n\to\infty}\int |v-\bar{v}|^3\,f_n(v)\,f_n(\bar{v})\,dv\,d\bar{v}$$

and

$$\int v^2 \bar{f}(v)\, dv = \lim_{n \to \infty} \int v^2 f_n(v)\, dv$$

which implies that $\bar{\eta} = \eta(\bar{f})$.

The uniqueness of the minimum follows by the strict convexity of η. Moreover $\bar{f}$ is even by uniqueness, is decreasing for $v > 0$ by rearrangement arguments, and $\bar{f}(v) > 0$, because of the presence of the entropy term in η. Therefore, from (2.12), $\bar{f}$ results to be a solution of Eq. (2.5). Finally, being $\int d\bar{v}\, |v - \bar{v}|^3 f(\bar{v}) \in \mathbf{C}^3$ if $f \in \mathbf{L}_1$, by a bootstrap argument in (2.5) we conclude that $\bar{f} \in \mathbf{C}^\infty$. ∎

A remarkable property of the free energy functional is that it decreases along the solutions. Indeed if $f = f(v, t)$ is a classical solution with $f_0 \in \mathscr{P} \cap \mathbf{C}^2(\mathbb{R})$, and $\eta(f_0) < +\infty$, then $\eta(f(t)) \leqslant \eta(f_0)$ as follows by a direct computation and an integration by parts:

$$\begin{aligned}\dot{\eta} &= -\int \left[\log f(v) + \frac{\lambda}{3\sigma} \int |v - \bar{v}|^3 f(\bar{v})\, d\bar{v} + \frac{\beta}{2\sigma} v^2 \right] \partial_v [(\lambda F - \beta v) f - \sigma\, \partial_v f]\, dv \\ &= -\frac{1}{\sigma} \int \frac{1}{f} |(\lambda F - \beta v) f - \sigma\, \partial_v f|^2\, dv \leqslant 0 \qquad (2.15)\end{aligned}$$

We are now in position to characterize the asymptotic behavior of the solution.

Theorem 2.2. For any $f_0 \in \mathscr{P} \cap \mathbf{C}^2(\mathbb{R})$, with $\int v^4 f_0 < +\infty$, and with $\eta(f_0) < +\infty$ we have:

$$\lim_{t \to \infty} \|f(t) - \bar{f}\|_{L_1} = 0 \qquad (2.16)$$

Proof. The proof is organized in two steps.

Step 1:

$$\lim_{t \to \infty} \eta(f(t)) = \bar{\eta} = \eta(\bar{f})$$

where $\bar{f}$ is the solution to (2.5) minimizing η.

Step 2:

$$\lim_{t \to \infty} \|f(t) - \bar{f}\|_{L_1} = 0$$

To prove step 1 we define $\bar{\eta} = \lim_{t \to \infty} \eta(f(t))$. Therefore:

$$\eta(f(t)) - \bar{\eta} = \int_t^\infty \dot{\eta}(f(s))\, ds$$

and there exists a diverging sequence $\{t_k\}_{k=1}^\infty$ such that $\dot{\eta}(f(t_k)) \to 0$. By the entropy and energy control we can find a probability density g such that the limit $k \to \infty$ of $f(t_k)$ is g in the sense of,the weak convergence of the measures (extracting subsequences if necessary).

Putting $f_k = f(t_k)$ and $F_k = -\phi * f_k$, we have:

$$\int |(\lambda F_k - \beta v)\, f_k - \sigma\, \partial_v f_k|\, dv = \int \frac{\sqrt{f_k}}{\sqrt{f_k}}\, |(\lambda F_k - \beta v)\, f_k - \sigma\, \partial_v f_k|\, dv$$
$$\leqslant (-\sigma \dot{\eta}(f_k))^{1/2} \to 0 \qquad (2.17)$$

As a consequence of (2.17), and (2.10) for the fourth moment of f_k, for any $\varphi \in \mathbf{C}_0^\infty$ we have

$$\int \varphi(v)(\lambda F_k - \beta v)\, f_k\, dv + \sigma \int \partial_v \varphi f_k\, dv$$
$$\to \int \varphi(v)(\lambda F_g - \beta v)\, g\, dv + \sigma \int \partial_v \varphi g\, dv = 0$$

where $F_g = -\phi * g$.

In particular g is a solution, in the sense of distribution, of the equation:

$$(\lambda F_g - \beta v)\, g + \sigma\, \partial_v g = 0 \qquad (2.18)$$

However, by a bootstrap argument, we realize that $g \in \mathbf{C}^\infty$ so that g solves (2.18) classically. Finally, since g is positive, we conclude that it is also a solution to Eq. (2.5). Therefore g is a stationary point of η, and by the strict convexity of η we conclude that $g = \bar{f}$.

We now show that f_k is a sequence uniformly bounded in $\mathbf{W}^{1,1}$. Indeed:

$$\|\partial_v f_k\|_{L_1} \leqslant \frac{1}{\sigma} \int |(\lambda F_k - \beta v)\, f_k - \sigma\, \partial_v f_k|\, dv + \frac{1}{\sigma} \int |(\lambda F_k - \beta v)\, f_k|\, dv \qquad (2.19)$$

The first term is uniformly bounded by $\sigma^{-1/2}\, |\dot{\eta}(f_k)|^{1/2}$ which is vanishing, while the second one is uniformly bounded by virtue of (2.10).

As a consequence, being the kinetic energy uniformly bounded, f_k is strongly converging in $\mathbf{L}_1$ and also a.e. converging (passing to subsequences, if necessary).

We now want to show the convergence of the entropies. By the $\mathbf{W}^{1,1}$ bound, we have:

$$\sup_k \|f_k\|_{L_\infty} \leqslant C \tag{2.20}$$

so that, by the dominated convergence theorem:

$$\int_{|v| \leqslant R} f_k \log f_k \to \int_{|v| \leqslant R} \bar{f} \log \bar{f} \tag{2.21}$$

Moreover we obtain the bound:

$$\left| \int_{|v| > R} f_k \log f_k \right| \leqslant O\left(\frac{1}{R}\right) \tag{2.22}$$

In fact, by (2.20) and the energy bound:

$$\int_{|v| > R} f_k \log f_k \chi(f_k > 1) \leqslant C \int_{|v| > R} f_k \leqslant \frac{C}{R^2} \tag{2.23}$$

Moreover, proceeding as in (2.14):

$$\int_{|v| > R} f_k \log f_k \chi(f_k < 1) \leqslant \frac{\beta}{\sigma} \int_{|v| > R} |v| \, f(v) \, dv + M \int_{|v| > R} e^{-\beta/2\sigma |v|} \, dv \tag{2.24}$$

Therefore

$$\int f_k \log f_k \to \int \bar{f} \log \bar{f} \tag{2.25}$$

The other terms of η are continuous in f if the fourth moment is bounded, so that:

$$\lim_{t \to \infty} \eta(f(t)) = \lim_{k \to \infty} \eta(f_k) = \bar{\eta} = \eta(\bar{f}) \tag{2.26}$$

This achieves the proof of step 1.

The second step follows by the inequality:

$$\eta(f) - \eta(\bar{f}) \geq \frac{1}{2} \int \frac{(f-\bar{f})^2}{f+\bar{f}} \tag{2.27}$$

which is an easy consequence of the analysis of the second derivative of η.

By the Schwartz inequality we have

$$\|f-\bar{f}\|_{L_1} \leq \left(\int \frac{(f-\bar{f})^2}{f+\bar{f}} \right)^{1/2} \left(\int (f+\bar{f}) \right)^{1/2} \tag{2.28}$$

and thus:

$$\|f(t)-\bar{f}\|_{L_1} \leq 2(\eta(f(t)) - \eta(\bar{f}))^{1/2} \to 0 \tag{2.29}$$

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